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A quasi-variational approach to a competitive economic equilibrium problem without strong monotonicity assumption

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Abstract This paper is focused on the investigation of the Walrasian economic equilibrium problem involving utility functions. The equilibrium problem is here reformulated by means of a quasi-variational inequality problem. Our goal is to give an existence result without assuming strong monotonicity conditions. To this end, we make use of a perturbation procedure. In particular, we will consider suitable perturbed utility functions whose gradient satisfies a strong monotonicity condition and whose associated equilibrium problem admits a solution. Then, we will prove that the limit solution solves the unperturbed problem. We stress out that our result allows us to consider a wide class of utility functions in which the Walrasian equilibrium problem may be solved.

Keywords Walrasian competitive equilibrium · Quasi-variational inequality

Mathematics Subject Classification (2000) 58E17 · 58E35

1 Introduction

In this paper, a competitive economic equilibrium problem involving utility functions is considered. This problem is a particular case of a general equilibrium problem (see e.g. [3,4,10,11,15,16]). The fixed point theory has been successfully applied by some authors to establish the existence of the Walrasian competitive equilibrium (see e.g. [1,18,19]). Recently, an alternative approach to the study of general economic equilibria has been

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performed by a suitable equivalent variational formulation. In such way also the Walrasian equilibrium problem can be studied by means of the variational inequality theory as it happens for several equilibrium problems (see e.g. [2,5–7,12,13,15]). We would like to remark that the Walrasian equilibrium problem can be considered from a dynamical point of view by using an evolutionary variational formulation (see e.g. [8,9]).

In particular, in [7] the authors reformulated a Walrasian competitive equilibrium problem for a pure exchange economy in terms of a quasi-variational inequality and developed useful techniques to establish the existence of a Walrasian equilibrium under strong monotonicity assumptions on the gradient of the utility functions. Now, the main purpose of this note is to obtain an existence result of equilibrium under weaker assumption than strong monotonicity. Precisely, the gradient of the utility functions will be assumed monotone only.

It is well-known that, even if somewhat restrictive, the strong monotonicity assumption is a key point in most of the approaches and, in particular, when unbounded sets are involved. We will be able, as said before, to relax the strong monotonicity condition thanks to a perturbation argument on the utility functions. From an economic point of view, the advantage of assuming monotonicity instead of strong monotonicity is that we can take into account a larger class of utility functions (frequently employed in the economic literature) which includes, for instance, functions non decreasing along any direction.

The plan of the paper is the following. We firstly present the Walrasian equilibrium problem and introduce some notations. Section 2 is devoted to the study of equilibrium by using a variational approach, i.e., the equilibrium is characterized in terms of a quasi-variational inequality. By perturbing suitably the utility functions, we will be able to apply the existence theorem obtained in [7]. This allows us to get a solution for the perturbed problem. Then, we will show that this solution converges to a solution of the unperturbed problem as the perturbation term tends to 0. Finally, Sect. 3 is dedicated to conclusions.

Let us consider a marketplace consisting of l different goods indexed by j and n agents indexed by a; let e_a^j and x_a^j denote, respectively, the nonnegative endowment and consumption by agent a relative to the commodity j. Then, $e_a = (e_a^1, e_a^2, \dots, e_a^l) \in \mathbb{R}_+^l$ and $x_a = (x_a^1, x_a^2, \dots, x_a^l) \in \mathbb{R}_+^l$ are the initial endowment vector and the consumption choice vector of agent a and $x \equiv (x_1, x_2, \dots, x_n)^T \in \mathbb{R}_+^{n \times l}$ is the consumption of market. We suppose that each agent a is endowed at least with a positive quantity of commodity (survivability assumption):

$$\forall a = 1, \dots, n \quad \exists j : \quad e_a^j > 0.$$

Let p^j denote the nonnegative price associated to the commodity j and

$$p = (p^1, p^2, \dots, p^l) \in P := \left\{ p \in \mathbb{R}_+^l : \sum_{j=1}^{l} p^j = 1 \right\}$$

the price vector. In this market, where a competitive behavior prevails, the agents' preferences, relative to the consumption x, are expressed by a utility function $u_a : \mathbb{R}_+^l \to \mathbb{R}$. We suppose that utility functions are concave and $C^1(\mathbb{R}_+^l)$.

In this market, the aim of each of the agents is to maximize his utility by performing pure exchange of the given goods. There are natural constraints that the consumers must satisfy: the wealth of a consumer is his initial endowment and the total amount of goods that a consumer can acquire or buy is at most equal to his initial wealth, i.e., the goods that the consumer sells off. This means that, for all $a = 1, \dots, n$ and for all $p \in P$, we have the following maximization problem

$$find \quad \overline{x}_a \in M_a(p) \quad such\ that \quad u_a(\overline{x}_a) = \max_{x_a \in M_a(p)} u_a(x_a), \tag{1}$$

where

$$M_a(p) = \left\{ x_a \in \mathbb{R}^l : \ x_a^j \geq 0 \ \forall j = 1, \ldots, l, \ \sum_{j=1}^{l} p^j (x_a^j - e_a^j) \leq 0 \right\}.$$

We denote by $\Gamma_a(p)$ the solution set to maximization problem (1). Of course $\Gamma_a(p)$ is a closed and convex set of $\mathbb{R}_+^l$ for each $a = 1, \ldots, n$ and $p \in P$ such that $\Gamma_a(p) \neq \emptyset$.

Moreover, for each $j = 1, \ldots, l$, we consider the total amount θ^j of the initial endowment relative to the commodity j, that is $\theta^j = \sum_{a=1}^n e_a^j$, and define the following set

$$K_a = \bigcup_{j=1}^{l} K_a^j$$

where

$$K_a^j = \left\{ x \in \mathbb{R}_+^l : x^j \leq e_a^j \right\} \cap \prod_{i=1}^{l} [0, \theta^i].$$

Since in this market there is not production, for each commodity, the choices must be such that the total consumption doesn't exceed the total endowment:

$$\sum_{a=1}^{n} \left(x_a^j - e\,_a^j \right) \leq 0.$$

Then, the competitive equilibrium conditions of a pure exchange economic market take the following form:

Definition 1 Let $\overline{p} \in P$ and let $\overline{x} \in M(\overline{p}) := \Pi_{a=1}^n M_a(\overline{p})$. We say that the pair $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ is a competitive equilibrium if and only if:

for all $a = 1, \ldots, n$

$$u_a(\overline{x}_a) = \max_{x_a \in M_a(\overline{p})} u_a(x_a), \tag{2}$$

and for all $j = 1, 2, \ldots, l$:

$$\sum_{a=1}^{n} (\overline{x}_a^j - e\,_a^j) \leq 0. \tag{3}$$

We will see that, under suitable assumptions, the market is regulated by Walras' law, that is for all competitive equilibrium $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ we have

$$\sum_{j=1}^{l} \overline{p}^j (\overline{x}_a^j - e_a^j) = 0\,, \tag{4}$$

for all $a = 1, \ldots, n$.

Condition (4) states that the amount that the agent a pays for acquiring the goods that maximize his utility, $\sum_{j=1}^l \overline{p}^j \overline{x}_a^j$, is equal to the amount that the agent receives for his initial endowment $\sum_{j=1}^l \overline{p}^j e_a^j$.

2 Variational approach

In this section we are concerned with the study of competitive equilibrium by using a variational approach. Firstly, we prove the following result.

Proposition 1 *Let* $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ *be a competitive equilibrium; then* $(\overline{p}, \overline{x}) \in P \times \prod_{a=1}^{n} K_a$.

Proof By condition (3) one has $\overline{x}_a \in \prod_{i=1}^{l}[0, \theta^i]$. Moreover, we have that $\overline{x}_a \in K_a$; in fact if it was $\overline{x}_a^j > e_a^j$ for all $j = 1, \ldots, l$, we should have

$$\sum_{j=1}^{l} \overline{p}^j \left(\overline{x}_a^j - e_a^j\right) > 0$$

and then $\overline{x}_a \notin M_a(\overline{p})$. Hence, there exists $j \in \{1, \ldots, l\}$ such that $\overline{x}_a^j \leq e_a^j$; this implies $\overline{x}_a \in K_a$. □

The next proposition states the conditions under which the market is regulated by the Walras' law.

Proposition 2 *Assume that the utility functions* u_a *are concave and belonging to* $C^1(\mathbb{R}_+^l)$ *and that the following conditions hold:*

$$\begin{aligned} &\nabla u_a(x_a) \neq 0 \quad && \forall x_a \in K_a; \\ &\frac{\partial u_a(x_a)}{\partial x_a^j} > 0 \quad && \forall x_a \in K_a \text{ and } \forall j = 1, \ldots, l \text{ such that } x_a^j = 0. \end{aligned} \tag{5}$$

Let $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ *be a competitive equilibrium. Then, condition* (4) *holds.*

Proof Arguing by contradiction, we assume that there exists an index $a \in \{1, \ldots, n\}$ such that

$$\sum_{j=1}^{l} \overline{p}^j \left(\overline{x}_a^j - e_a^j\right) < 0. \tag{6}$$

We note that condition (2) and the second assumption of (5) imply $\overline{x}_a^j > 0$ for all $j = 1, \ldots, l$ which, jointly to (6), yields $\overline{x}_a \in int(M_a(\overline{p}))$. Consequently, since $\overline{x}_a \in \Gamma_a(\overline{p})$ we get $\nabla u_a(\overline{x}_a) = 0$ which contradicts the first of (5). □

The following theorem establishes a connection between competitive equilibria and the solutions of a quasi-variational inequality.

Theorem 1 *Let* u_a *be concave and* C^1. *If* $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ *is a solution to the quasi-variational inequality:*

$$\left\langle \sum_{a=1}^{n} \overline{x}_a - e_a, p - \overline{p} \right\rangle + \sum_{a=1}^{n} \langle \nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle \leq 0 \quad \forall\, (p, x) \in P \times M(\overline{p}), \tag{7}$$

then $(\overline{p}, \overline{x})$ *is a competitive equilibrium. Conversely, assume that also conditions* (5) *hold. Then, any competitive equilibrium* $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ *is solution of* (7).

Proof Firstly, we make the following two remarks:

(a) $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ is a solution to quasi-variational inequality (7) if and only if $\overline{p}$ is a solution to:

$$\left\langle \sum_{a=1}^{n} (\overline{x}_a - e_a), p - \overline{p} \right\rangle \leq 0 \quad \forall p \in P. \tag{8}$$

and, for all $a = 1, \dots, n$, $\overline{x}_a$ is a solution to:

$$\langle \nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle \leq 0 \quad \forall x_a \in M_a(\overline{p}). \tag{9}$$

This is easily seen by testing equation (7), respectively, with $(p, \overline{x})$, $p \in P$, and $(\overline{p}, x)$, $x \in M(\overline{p})$.

(b) Since $u_a \in C^1(\mathbb{R}^l_+)$, it is well known that $\overline{x}_a$ is a maximum point of u_a in $M_a(\overline{p})$ if and only $\overline{x}_a$ is a solution to variational inequality (9).

Now, let $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ be a solution to quasi-variational inequality (7). Then, by remark (a), variational inequality (8) holds and since $\overline{x} \in M(\overline{p})$ we have

$$\left\langle \sum_{a=1}^{n} (\overline{x}_a - e_a), p \right\rangle = \left\langle \sum_{a=1}^{n} (\overline{x}_a - e_a), p - \overline{p} \right\rangle + \left\langle \sum_{a=1}^{n} (\overline{x}_a - e_a), \overline{p} \right\rangle$$
$$\leq \sum_{a=1}^{n} \langle \overline{x}_a - e_a, \overline{p} \rangle \leq 0 \quad \forall p \in P.$$

In particular, selecting $p = (0, \dots, 0, 1, 0, \dots, 0)$, with 1 at the j-th position, we get equilibrium condition (3). Moreover, by remarks (a) and (b), we also have that $\overline{x}_a$ is a maximum point of u_a in $M_a(\overline{p})$. Hence $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ is a competitive equilibrium of a pure exchange economic market.

Conversely, assume that also conditions (5) hold and suppose that $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ is a competitive equilibrium. Then, by Proposition 2 we have that the Walras' law (4) holds. Thus, using (4) and equilibrium condition (3), we have:

$$\left\langle \sum_{a=1}^{n} (\overline{x}_a - e_a), p - \overline{p} \right\rangle = \left\langle \sum_{a=1}^{n} (\overline{x}_a - e_a), p \right\rangle \leq 0 \quad \forall p \in P.$$

Therefore, $\overline{p}$ is a solution to variational inequality (8). Moreover, by remark b), $\overline{x}_a$ is a solution to variational inequality (9). Hence by remark (a), $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ is a solution to quasi-variational inequality (7). □

For the sequel it will be useful to recall the statement of Theorem 4 of [7]:

Theorem 2 *Let $u_a \in C^1(\mathbb{R}^l_+)$ be such that $-\nabla u_a$ is a strongly monotone operator $\forall a = 1, \dots, n$. Then, there exists a unique solution $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ to quasi-variational inequality* (7).

Note that in the statement of the above theorem we only require the strong monotonicity of the operator $-\nabla u_a$, while in [7] conditions (5) are throughout assumed on the utility functions. However, we stress out that these latter conditions are unnecessary for the validity of Theorem 2 as it is easy to check by reading the proof given in [7].

Now we would like to give a preliminary existence result for the competitive equilibrium problem by introducing perturbed utility functions. For each positive number ε we define the utility functions

$$u_{a,\varepsilon}(x_a) = u_a(x_a) - \varepsilon\|x_a\|^2 \quad \forall\, x_a \in \mathbb{R}^l_+.$$

and We are able to establish the following result

Theorem 3 *Let u_a be concave and C^1 in $\mathbb{R}^l_+$ for all $a = 1, \ldots, n$. Then, there exists a unique solution $(\overline{p}_\varepsilon, \overline{x}_\varepsilon)$ to the following quasi-variational inequality*

$$\left\langle \sum_{a=1}^{n} (\overline{x}_{a,\varepsilon} - e_a),\, p - \overline{p}_\varepsilon \right\rangle + \sum_{a=1}^{n} \left\langle \nabla u_{a,\varepsilon}(\overline{x}_{a,\varepsilon}), x_a - \overline{x}_{a,\varepsilon} \right\rangle \leq 0 \quad for\ all\ (p, x) \in P \times M(\overline{p}_\varepsilon). \quad (10)$$

Proof Since u_a is concave and C^1 in $\mathbb{R}^l_+$, then, of course, $u_{a,\varepsilon}$ belongs to $C^1(\mathbb{R}^l_+)$. Moreover one has that $-\nabla u_{a,\varepsilon}$ is strongly monotone:

$$\langle -\nabla u_{a,\varepsilon}(x) + \nabla u_{a,\varepsilon}(y), x - y\rangle \geq 2\varepsilon\|x - y\|^2, \quad (11)$$

for all $a = 1, \ldots, n$ and for all $x,\ y \in \mathbb{R}^l_+$. In fact, following standard arguments, since $-u_a$ is convex and C^1 in $\mathbb{R}^l_+$, one has (see for instance Theorem 25.1 of [17])

$$-u_a(x) \geq -u_a(y) + \langle -\nabla u_a(y), x - y\rangle \quad \text{for every } \ x, y \in \mathbb{R}^l_+.$$

Interchanging the roles of x and y we have

$$-u_a(y) \geq -u_a(x) + \langle -\nabla u_a(x), y - x\rangle \quad \text{for every } \ x, y \in \mathbb{R}^l_+.$$

Finally, adding side to side we get

$$\langle -\nabla u_a(x) + \nabla u_a(y), x - y\rangle \geq 0 \quad \text{for every } \ x, y \in \mathbb{R}^l_+.$$

Consequently,

$$\begin{aligned}\left\langle -\nabla u_{a,\varepsilon}(x) + \nabla u_{a,\varepsilon}(y), x - y\right\rangle &= \langle -\nabla u_a(x) + 2\varepsilon x + \nabla u_a(y) - 2\varepsilon y, x - y\rangle \\ &= \langle -\nabla u_a(x) + \nabla u_a(y), x - y\rangle \\ &\quad +2\varepsilon\|x - y\|^2 \geq 2\varepsilon\|x - y\|^2,\end{aligned}$$

then (11) holds. Hence, by applying Theorem 2, there exists a unique solution to quasi-variational inequality (10). □

In order to obtain our main result, we need to recall the concept of set convergence in the sense of Mosco:

Definition 2 ([14]) Let $(V, \|\cdot\|)$ be an Hilbert space and let $\mathbf{K} \subset V$ be a closed, nonempty, convex set. A sequence of nonempty, closed, convex sets $\mathbf{K}_n$ converges to $\mathbf{K}$ as $n \to +\infty$, i.e. $\mathbf{K}_n \to \mathbf{K}$, if and only if

- (M_1) for any $H \in \mathbf{K}$, there exists a sequence $\{H_n\}_{n\in \mathrm{N}}$ strongly converging to H in V such that H_n lies in $\mathbf{K}_n$ for all n,
- (M_2) for any $\{H_{k_n}\}_{n\in \mathrm{N}}$ weakly converging to H in V, such that H_{k_n} lies in $\mathbf{K}_{k_n}$ for all n, then the weak limit H belongs to $\mathbf{K}$.

Proposition 3 *Let $\{p_n\} \subseteq P$ be such that $p_n \to p$, then the sequence of sets $M_a(p_n)$ converges to $M_a(p)$ in Mosco's sense.*

Proof See e.g. [7]. □

Finally, we give our main result.

Theorem 4 *Let u_a be concave and C^1 in $\mathbb{R}^l_+ \forall a = 1, \ldots, n$. Then, there exists at least one competitive equilibrium $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$.*

Proof Thanks to Theorem 1 it is sufficient to prove that there exists at least one solution to quasi-variational inequality (7). We prove the theorem by means of the following steps.

1) Let $\{\varepsilon_k\}$ be a sequence of positive real numbers such that $\lim_{k\to+\infty} \varepsilon_k = 0$ and put for simplicity $u_{a,k} = u_{a,\varepsilon_k}$. By Theorem 3, for all $k \in \mathbb{N}$, there exists a solution $(\overline{p}_k, \overline{x}_k)$ to quasi-variational inequality (10) with $\varepsilon = \varepsilon_k$. We prove that the sequence $(\overline{p}_k, \overline{x}_k)$ converges to some $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$. To this end, firstly we note that, by Theorem 1 $(\overline{p}_k, \overline{x}_k)$ is a competitive equilibrium relative to utility $u_{a,k}$. Then, by Proposition 1 one has $(\overline{p}_k, \overline{x}_k) \in P \times \prod_{a=1}^n K_a$ for all $k \in \mathbb{N}$. Since the set $P \times \prod_{a=1}^n K_a$ is compact, without loss of generality, we can suppose $\{(\overline{p}_k, \overline{x}_k)\}$ converging to some $(\overline{p}, \overline{x}) \in P \times \prod_{a=1}^n K_a$.
Therefore

$$\lim_{k\to\infty} \overline{x}_{a,k} = \overline{x}_a \quad \forall a = 1, \ldots, n, \quad \text{and} \tag{12}$$

$$\lim_{k\to\infty} \overline{p}_k = \overline{p}. \tag{13}$$

By (13) one has that the sequence of sets $\{M_a(\overline{p}_k)\}$ converges to $M_a(\overline{p})$ in Mosco's sense. Then, from condition (M_2) it follows that $\overline{x}_a \in M_a(\overline{p})$.

2) Now, we prove that the limit $(\overline{p}, \overline{x}_a)$ is a solution to quasi-variational inequality (7). Since $u_a \in C^1(\mathbb{R}^l_+)$, from (12) one has:

$$\lim_{k\to\infty} \nabla u_a(\overline{x}_{a,k}) = \nabla u_a(\overline{x}_a).$$

Then, taking into account that the sequences $\{\overline{x}_{a,k}\}$ are bounded, it follows

$$\lim_{k\to\infty} \nabla u_{a,k}(\overline{x}_{a,k}) = \lim_{k\to\infty} (\nabla u_a(\overline{x}_{a,k}) - 2\varepsilon_k \overline{x}_{a,k}) = \nabla u_a(\overline{x}_a). \tag{14}$$

From (M_1), for all $x_a \in M_a(\overline{p})$ there exists a sequence $\{y_{a,k}\}$ such that $y_{a,k} \in M_a(\overline{p}_k)$ and

$$\lim_{k\to\infty} y_{a,k} = x_a. \tag{15}$$

Being $(\overline{p}_k, \overline{x}_{a,k})$ solution to quasi-variational inequality (10) it follows:

$$\left\langle \sum_{a=1}^n (\overline{x}_{a,k} - e_a), p - \overline{p}_k \right\rangle + \sum_{a=1}^n \left\langle \nabla u_{a,k}(\overline{x}_{a,k}), y_{a,k} - \overline{x}_{a,k} \right\rangle \le 0, \quad \forall p \in P.$$

Passing to the limit as $k \to \infty$ from conditions (12), (13), (14), (15) one has that $(\overline{p}, \overline{x})$ is a solution to quasi-variational inequality (7).

□

3 Conclusions

In [7], under strong monotonicity assumptions on the gradient of utility functions, a demand function $x_a(p)$ was defined and the competitive equilibrium was characterized as a solution to a quasi-variational inequality. In this paper, weaker assumptions on utility functions are considered.

In this framework, we can see that the demand of the agent a is, actually, a multi-valued map:

$$p \to \Gamma_a(p) = \left\{ x_a \in \mathbb{R}^l_+ \ : \ x_a \text{ solution to maximization problem } (1) \right\}.$$

The equilibrium is, in this case, again characterized as a solution to a quasi-variational inequality.

The main purpose of this paper has been to obtain the existence of the competitive equilibrium under weaker assumptions on utility functions, namely we have assumed u_a concave and of class $C^1(\mathbb{R}^l_+)$ or, in other words, ∇u_a monotone. To this aim, we take advantage of a previous existence result, given in [7], applied to perturbations of utility functions. By means of this more general mathematical formulation we have been able to study a quasi-variational inequality involving utility functions with no strong monotonicity assumption.

Thus, the existence of equilibrium has been given for a larger class of utility functions.

Finally, we want to stress out that, in our assumptions, the excess demand operator $\sum_{a=1}^{n} \langle \bar{x}_a - e_a, \cdot \rangle$ is not a single valued map. This means that, if we want to get our existence result (Theorem 4) following the arguments proposed in [7], we are led to study the following *generalized* variational inequality

$$find \quad \bar{p} \in P \quad and \quad \bar{x}_a \in \Gamma_a(\bar{p}) \quad such \quad that \quad \sum_{a=1}^{n} \langle \bar{x}_a - e_a, p - \bar{p} \rangle \leq 0 \quad \forall p \in P.$$

We have been able to avoid of doing this just by applying the perturbation method introduced in this paper.

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